Games, graphs, and machines

Relations

July 26, 2024

Reminders

1. Join zulip!
2. Workshops start next week.
3. Homework 1 due next Friday.
4. Reflective check-in due Monday.

The number of relations

Suppose $A = \{1, 2, 3\}$ and $B = \{1, 2, 3, 4\}$. How many relations are there between A and B?

2^{64} 2^{12} 12 $2^{12}-1$. $\emptyset$ is a valid relation!

Relation = Subset of $\underbrace{A \times B}_{12 \text{ elts.}}$

2^{12}

$\{\text{Relations}\} = \text{Pow}(A \times B)$

Reflexive relations

We say that a relation $R \subset S \times S$ is *reflexive* if for all $s \in S$, we have $(s, s) \in R$

Are the following relations reflexive?

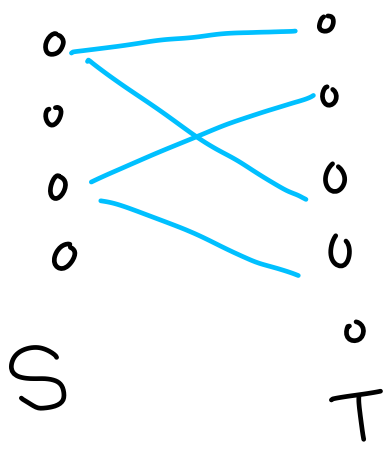
1. $\leq$ on $\mathbb{R}$ ✓
2. $<$ on $\mathbb{R}$ ✗
3. the relation R on $\mathbb{Z}$ defined by $(a, b) \in R$ if 2 divides $a + b$. ✓
4. Same as above, but with 2 replaced by 3. ✗

e.g. $(1,1) \notin R$.

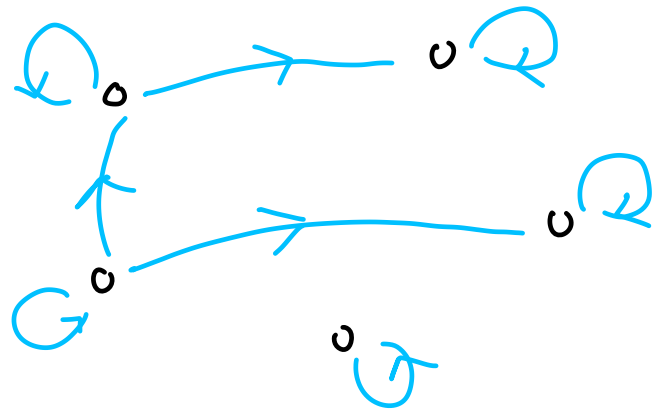
5. Replace $a+b$ by $a-b$ ✓

Visualising relations

Rel. on S and T



Rel on S (T = S)



Directed graph.

Loop at every vertex
= Reflexive

Symmetric relations

We say that a relation $R \subset S \times S$ is *symmetric* if for all $s \in S$ and $t \in S$, if $(s, t) \in R$ then $(t, s) \in R$.

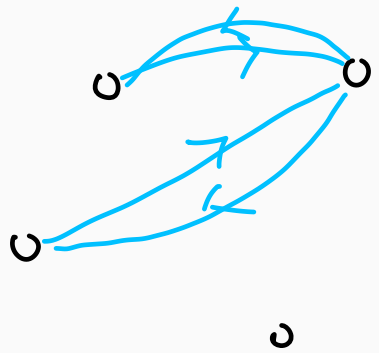
Find a relation on $\mathbb{Z}$ that is symmetric and one that is not symmetric.

Symmetric: $=$, 2 divides the diff.

Not sym: $<$, $\leq$, divides

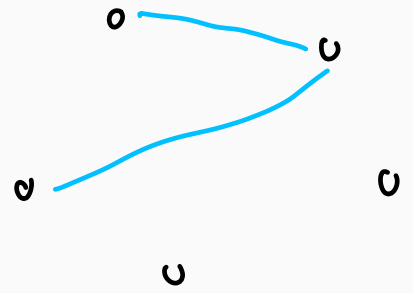
Sym but not reflexive: 2 does not divide diff

Picture



Simplify ⇝

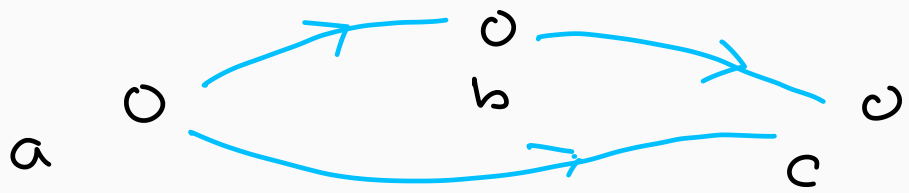
Undirected graph →



Transitive relations

We say that a relation $R \subset S \times S$ is *transitive* if for all $a, b, c \in S$ if $(a, b) \in R$ and $(b, c) \in R$ then $(a, c) \in R$.

Find a relation on $\mathbb{R}$ that is transitive and one that is not transitive.



Ex: $<$, $>$, $\leq$, $\geq$, divisibility

Non-ex: (a,b) if $a+b$ div by 3

$(1,2) \in R$, $(2,4) \in R$, $(1,4) \notin R$

Transitive relations (continued)

Are the following relations transitive?

1. $\leq$ on $\mathbb{R}$
2. $<$ on $\mathbb{R}$
3. the relation R on $\mathbb{Z}$ defined by $(a, b) \in \mathbb{Z}$ if 2 divides $a + b$.
4. Same as above, but with 2 replaced by 3.

Input/Output relation

Consider $R \subset \mathbb{R} \times \mathbb{R}$ defined by

$$R = \{(x, y) \mid x^3 - xy + x - 1 = 0\}.$$

Is R the input/output relation of a function $f : \mathbb{R} \to \mathbb{R}$?

To be I/O of a func.

every x must have a

unique y s.t. $(x,y) \in R$

$x = 0$ has no y. ← so no.